Quiver Varieties

HIRAKU NAKAJIMA
(Kavli IPMU, The University of Tokyo)

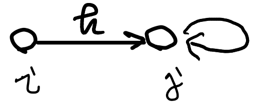
2024-07-25 Physics Lecture

References: Kirillov Quiver representations and quiver varieties, AMS
Ginzburg Lectures on Nakajima's quiver varieties

PART 1. Definition and Basic Properties

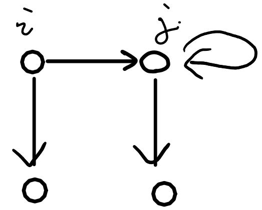
$Q = (Q_0, Q_1)$: quiver = finite oriented graph

Q_0: vertices, Q_1: edges

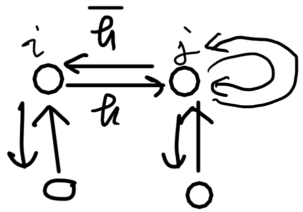


$o(h) = i$
$i(h) = j$

Q^f = framed quiver



$(Q^f)^\#$ = doubled framed quiver
add opposite edges



$V = \bigoplus_i V_i$ $\quad$ Q_0 – graded (cpx) vector space, dim $< \infty$
$W = \bigoplus_i W_i$

$$\mathbb{M}(V,W) := \bigoplus_{h \in Q_1 \sqcup \overline{Q_1}} \underset{\substack{\Downarrow \\ B_h}}{\mathrm{Hom}(V_{o(h)}, V_{i(h)})} \oplus \bigoplus_{i \in Q_0} \underset{\substack{\Downarrow \\ a_i}}{\mathrm{Hom}(W_i, V_i)} \oplus \underset{\substack{\Downarrow \\ b_i}}{\mathrm{Hom}(V_i, W_i)}$$

(or $\mathbb{M}$)

$$= T^* \underline{\mathbb{N}(V,W)} \qquad \bigoplus_{h \in Q_1} \underline{\qquad\qquad} \oplus \bigoplus$$

$\therefore$ symplectic vector space

group actions

- $G = \prod GL(V_i) \curvearrowright \mathbb{N}(V,W),\ \mathbb{M}(V,W)$ by conjugation
- $G_W = \prod GL(W_i) \curvearrowright$

These actions preserve the symplectic form ω

- $\mathbb{C}^\times \curvearrowright \mathbb{M}(V,W) = \mathbb{N}(V,W) \oplus \underline{\mathbb{N}(V,W)^*}$

multiplication on the cotangent direction

"quotient" space of $\mathbb{M}(V,W)$ by G in symplectic geometry

$\mathbb{M}(V,W)/G$ is not symplectic (e.g. dimension could be odd)

$\in$
$[m]$

$$T_{[m]}(\mathbb{M}(V,W)/G) = T_m \mathbb{M}(V,W) \,/\, T_m(Gm)$$

$T_m\mathbb{M}(V,W)$ contains redundant directions $\perp^{\omega}$ $T_m G m$

Consider the simplest quiver $Q = \circ$ (no edge)

$$\begin{tikzcd} V \arrow[d, shift right, "b"'] \\ W \arrow[u, shift right, "a"'] \end{tikzcd}$$

$$\underset{\circlearrowleft}{M(V,W)} = \underset{\substack{\cup \\ a}}{\mathrm{Hom}(W,V)} \oplus \underset{\substack{\cup \\ b}}{\mathrm{Hom}(V,W)}$$

$$G = GL(V) \quad ; \quad (a,b) \mapsto (ga, bg^{-1})$$

Consider an open subset $\{ b \mid$ injective $\} / G$ of $\mathrm{Hom}(V,W)/G$.

$\mathrm{Image}\, b \subset W$ is a subspace with $\dim = \dim V$

$\{ b \mid \text{injective} \} / G$ = Grassmannian manifold of $\dim = \dim V$ subspaces

and $T_{[b]}\mathrm{Grass} \cong \mathrm{Hom}(V,W)/b\cdot\mathrm{Hom}(V,V) \cong \mathrm{Hom}(V, W/\mathrm{Im}\, b)$

The dual space is $\mathrm{Hom}(V, W/\mathrm{Im}\, b)^* \cong \mathrm{Hom}(W/\mathrm{Im}\, b, V) \subset \mathrm{Hom}(W,V)$.

difference is the redundant directions

$\displaystyle\bigcup_{\mathrm{Im}\, b \in \mathrm{Grass}} \mathrm{Hom}(W/\mathrm{Im}\, b, \underset{(\cong V)}{\mathrm{Im}\, b})$ is the cotangent bundle of the Grassmannian manifold. $T^*\mathrm{Grass}$

It is a symplectic manifold.

Note $a \in \mathrm{Hom}(W, V)$ is contained in $\mathrm{Hom}(W/\mathrm{Im}\, b, V)$

$$\Longleftrightarrow \quad ab = 0$$

☆ This consideration leads us to introduce the moment map and the symplectic reduction

$$\mu : \mathbb{M}(V, W) \longrightarrow \mathrm{Lie}\,\mathbb{G}^* \underset{\text{trace}}{\cong} \mathrm{Lie}\,\mathbb{G} \; ; \; (B_h, a_i, b_i) \longmapsto \Big(\sum \underbrace{\varepsilon(h)}_{\pm 1} B_h B_{\bar{h}} + a_i b_i\Big)_i$$

moment map

$\therefore$ $\mathrm{Ker}\, d\mu$ should be a correct subspace.

codim $= \dim \mathbb{G}$ if $d\mu$ is full rank

$\mu^{-1}(0)/\mathbb{G}$ or more generally $\mu^{-1}(\bigoplus_i \zeta_i \, \mathrm{id}_{V_i})/\mathbb{G}$ $\quad (\zeta_i \in \mathbb{C})$

symplectic reduction

is the "quotient" space in symplectic geometry.

However, $\mu^{-1}(0)/G$ is not yet a "correct" quotient space, as illustrated in the example of $T^*\mathrm{Grass}$.

We only considered $\{b : \text{injective}\}$.

This issue already appears when we consider the quotient space

$$\mathrm{Hom}(V, W)/GL(V).$$

Quotient spaces are non-Hausdorff / not manifold in general.

There are several approaches:

1. extend the notion of spaces —— we don't adopt this today
2. restrict to open subsets like b: injective (GIT quotient)
3. replace quotient spaces by sets of closed G-orbits. (categorical quotient)

Return back to the example $Q = \circ$ $\begin{array}{c} V \\ b\downarrow\uparrow a \\ W \end{array}$

We considered b: injective for the 2nd approach.

3rd approach?. e.g. extreme case $b=0$?

$$g(t) = t\cdot \mathrm{id} \qquad (g(t)a, \underset{\overset{\shortparallel}{0}}{b}g(t)^{-1}) = (g(t)a, 0) \xrightarrow[t\to 0]{} (0,0)$$

$\therefore$ $a=0$ is the only closed orbit with $b=0$.

More generally, $\left\{ \begin{array}{c}\text{closed } GL(V)\text{-orbits} \\ \text{in } M(V,W)\end{array} \right\} \hookrightarrow \mathrm{End}(W)$ fundamental thm in invariant theory

$$[a,b] \longmapsto X = ba$$

Note $X=0$ if $b=0$

$(a,b) \in \mu^{-1}(0)$ (i.e. $ab=0$) $\Longrightarrow X^2 = ba\cdot ba = 0$

<u>Th</u>. $\left\{ \begin{array}{c}\text{closed } GL(V)\text{-orbits} \\ \text{in } \mu^{-1}(0)\end{array} \right\} = \left\{ X \in \mathrm{End}(W) \;\middle|\; \begin{array}{l} X^2 = 0 \\ \mathrm{rank}\, X \leq \dim V \end{array} \right\}$

Let us consider the two approaches for general quiver Q.
(unfortunately, the GIT quotient might look artificial)

<u>Definitions</u>

(1) $(B, a, b) \in \mu^{-1}(0)$ (more generally $\in \mathbb{M}(V, W)$) is stable

$\overset{\text{def.}}{\Longleftrightarrow}$ $0 \neq \nexists S \subset V$ Q_0-graded subspace s.t. $\begin{array}{l} B(S) \subset S \\ S \subset \operatorname{Ker} b \end{array}$

$$\mu^{-1}(0)^{st} = \{\text{stable } (B, a, b)\} \underset{\text{open}}{\subset} \mu^{-1}(0)$$

(in Zariski / classical topology)

$$\mathfrak{M}(V, W) \overset{\text{def.}}{=} \mu^{-1}(0)^{st} / G$$

(2) $\mathfrak{M}_0(V, W) = \mu^{-1}(0) /\!\!/ G =$ the set of closed G-orbits in $\mu^{-1}(0)$

<u>Remark</u> For $Q = \emptyset$, stable $\Longleftrightarrow$ b : injective

Basic Fact from invariant theory

- $\overline{G\text{-orbit}}$ (closure) $=$ union of G-orbits
- unique G-orbit in $\overline{G\text{-orbit}}$ (closed G-orbit)

(remember $a=0$ for $b=0$)

$\therefore$ We have a natural map

$$\begin{array}{ccc} \mathcal{M}(V,W) & \xrightarrow{\pi} & \mathcal{M}_0(V,W) \\ \rotatebox{90}{$\in$} & & \\ Gm & \longmapsto & \text{closed } G\text{-orbit in } \overline{Gm} \end{array}$$

<u>Example</u>

$$\begin{array}{ccc} T^*\mathrm{Grass} = \bigcup_{\mathrm{Im}\, b} \mathrm{Hom}(W/_{\mathrm{Im}\, b}, \mathrm{Im}\, b) & \xrightarrow{\pi} & \{X \mid X^2=0,\ \mathrm{rk}\, X \leqq \dim V\} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ [a,b] & \longmapsto & X = ba \end{array}$$

This gives a resolution of singularities if $2\dim V \leqq \dim W$

say, if $\mathrm{rk}\, X = \dim V \Rightarrow \mathrm{Im}\, b = \mathrm{Im}\, X \quad \therefore \pi^{-1}(X) = \text{point}$

Basic properties of $\mathcal{M}, \mathcal{M}_0$

1) algebraic varieties (i.e., objects of algebraic geometry)
2) $\mathcal{M}$ is smooth. (cpx manifold)
3) $\pi : \mathcal{M} \longrightarrow \mathcal{M}_0$ is a projective morphism (in particular, proper $\pi^{-1}(\mathrm{pt}) : \mathrm{cpt}$)

Actions of $G_W = \prod GL(W_i)$ and $\mathbb{C}^\times$ descend to $\mathcal{M}$ and $\mathcal{M}_0$.

π is equivariant.

Example $Q = \circ \longrightarrow \circ$

$\dim V_1 = \dim V_2 = \dim W_1 = \dim W_2 = 1$

$$\begin{tikzcd} V_1 \arrow[r, shift left, "B_1"] \arrow[d, shift left, "b_1"] & V_2 \arrow[l, shift left, "B_2"] \arrow[d, shift left, "b_2"] \\ W_1 \arrow[u, shift left, "a_1"] & W_2 \arrow[u, shift left, "a_2"] \end{tikzcd}$$

$X \overset{\text{def}}{=} b_2 B_1 a_1$, $Y \overset{\text{def}}{=} b_1 B_2 a_2$, $Z \overset{\text{def}}{=} b_1 a_1 = B_1 B_2 = b_2 a_2$

$\mathcal{M}_0 \cong \{ XY = Z^3 \} \subset \mathbb{C}^3$

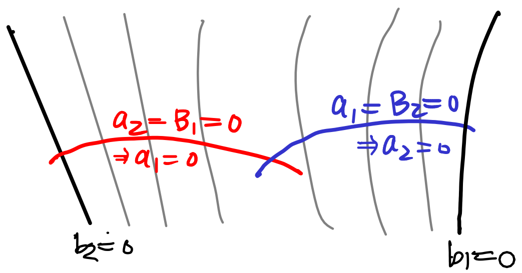
A_2 type singularity

$\mathcal{M} = \{ b_1 \neq 0 \} \cup \{ b_2 \neq 0 \}$ (by stability)

$\hookrightarrow$ normalize $b_1 = 1$. Then $\cong \begin{matrix} V_1 \rightleftarrows V_2 \\ \uparrow\downarrow \\ W_1 \end{matrix} / GL(V_2) \cong T^*\mathbb{P}^1$

$([b_2 : B_2],\ a_2 \oplus -B_1) : W_2 \oplus V_1 / \mathbb{C}(b_2 \oplus B_2) \longrightarrow V_2$

$b_2 = 1$, $\cong T^*\mathbb{P}^1$



$\mathbb{P}^1 \cap \mathbb{P}^1 =$ a point
$$\begin{tikzcd} V_1 \arrow[r, shift left, "0"] \arrow[d, shift left, "1"] & V_2 \arrow[l, shift left, "0"] \arrow[d, shift left, "1"] \\ \mathbb{C} \arrow[u, shift left, "0"] & \mathbb{C} \arrow[u, shift left, "0"] \end{tikzcd}$$

Theorem

Suppose Q has no edge loops.

Then (1) $\pi^{-1}(0)$ is a lagrangian subvariety of $\mathfrak{M}(V,W)$.
(singular in general)

(2) $\pi^{-1}(0)$ is homotopic to $\mathfrak{M}(V,W)$.

$\pi^{-1}(0)$ is denoted by $\mathcal{L}(V,W)$.

Ex (1) $T^*\mathrm{Grass} \supset \mathrm{Grass}$ (0 section)

(2) Ex in the previous page $\supset \mathbb{P}^1 \cup \mathbb{P}^1$

Remark Without assumption, $\dim \pi^{-1}(0) < \frac{1}{2} \dim \mathfrak{M}(V,W)$

$Q = \circ\!\circlearrowleft$ $\qquad$ $x \circlearrowright \mathbb{C} \circlearrowleft y$, $\quad \mathbb{C} \underset{0}{\overset{1}{\rightleftarrows}} \mathbb{C}$ $\qquad$ $\mathfrak{M} \cong \mathfrak{M}_0 \cong \mathbb{C}^2$ $\qquad$ π : isomorphism

$\therefore \pi^{-1}(0) = pt \subset \mathbb{C}^2$

PART 2. Relation to Representation theory of Kac-Moody Lie Algebras

We assume hereafter

Q : quiver $\underset{\substack{\text{forget orientation} \\ \text{of edges}}}{\rightsquigarrow}$ Dynkin diagram $\rightsquigarrow$ $\mathfrak{g} = \mathfrak{g}_Q$: (symmetric) Kac-Moody Lie algebra

C = Cartan matrix

$$C_{ij} = \begin{cases} 2 & (i=j) \\ \#\{ i \xrightarrow{h} j \} + \#\{ i \xleftarrow{h} j \} & (i \neq j) \end{cases}$$

We will construct 'integrable representations of $\mathfrak{g}$.

<u>Remark</u> $\mathfrak{g}$: symmetric $\overset{\text{def.}}{\Longleftrightarrow}$ $C_{ij} = C_{ji}$

<u>Open Problem</u>: Develop a theory of quiver varieties for C with $C_{ij} \neq C_{ji}$.

N-Weekes : Coulomb branches for nonsymmetric quivers

$\mathfrak{g}$ has a presentation by generators and relations.

- $\mathfrak{h}$: complex vector space with $\dim = \# Q_0 + \dim \operatorname{Ker} C$
- $\{\alpha_i\}_{i \in Q_0} \subset \mathfrak{h}^*$, $\{h_i\}_{i \in Q_0} \subset \mathfrak{h}$
  - linearly independent
  - $\langle h_i, \alpha_j \rangle = C_{ij}$

☆ e_i, f_i $(i \in Q_0)$, $h \in \mathfrak{h}$

- $[h, e_i] = \alpha_i(h) e_i$, $[h, f_i] = -\alpha_i(h) f_i$ $\quad h \in \mathfrak{h}$
- $[h_1, h_2] = 0$ $\quad h_1, h_2 \in \mathfrak{h}$
- $[e_i, f_j] = \delta_{ij} h_i$ $\qquad$ etc

GOAL : Construct operators e_i, f_i, h acting on (co)homology of $\mathfrak{M}(V, W)$ (more precisely $\bigoplus_V$)

so that they satisfy the above defining relations of $\mathfrak{g}$.

Example $Q = \circ$ $\qquad \mathfrak{g}_Q = \mathfrak{sl}_2$

$(r+1)$-dimensional irreducible representation :

$$\bullet \underset{e}{\overset{f}{\rightleftarrows}} \bullet \underset{e}{\overset{f}{\rightleftarrows}} \bullet \rightleftarrows \cdots\cdots \underset{e}{\overset{f}{\rightleftarrows}} \bullet \underset{e}{\overset{f}{\rightleftarrows}} \bullet$$ (up to scalar)

highest weight vector v_0

We use the integral structure given by $v_k := \dfrac{f^k}{k!} v_0$ ($v_0 =$ highest wt vector)

$$\begin{cases} h v_k = (r-2k) v_k \\ f v_k = (k+1) v_{k+1} \\ e v_k = (r-k+1) v_{k-1} \end{cases} \qquad 0 \leqq k \leqq r \qquad \left(v_{-1} = 0 = v_{r+1} \text{ by convention} \right.$$

$\mathfrak{M}(V, W) \cong T^* \mathrm{Grass}(k, r)$ $\qquad \mathrm{Grass}(k,r) = \{ k\text{-dim'l subspace in } \mathbb{C}^r \}$

k-dim $\quad$ r-dim $\quad$ nonempty $\Leftrightarrow 0 \leqq k \leqq r$ $\quad$ $k(r-k)$ dim'l cpx manifold

$$H_*(T^*\mathrm{Grass}(k,r)) \cong H_*(\mathrm{Grass}(k,r)) \supset H_{\underbrace{top}_{=2k(r-k)}}(\mathrm{Grass}(k,r)) = \mathbb{C}[\mathrm{Grass}(k,r)]$$

1-dim'l space

Want : $v_k = [\mathrm{Grass}(k,r)]$

In order to define e, f in geometric way, we use the technique of **Correspondences**.

Let us consider

$$\begin{tikzcd}[column sep=small] & \{ \overset{k-1}{S_1} \subset \overset{k}{S_2} \subset \mathbb{C}^r \} =: Q(k,r) \arrow[dl, "q_1"'] \arrow[dr, "q_2"] & \\ \mathrm{Grass}(k-1,r) & & \mathrm{Grass}(k,r) \end{tikzcd}$$

$$H_*(\mathrm{Grass}(k-1,r)) \underset{q_{1*}\, q_2^*}{\overset{q_{2*}\, q_1^*}{\rightleftarrows}} H_*(\mathrm{Grass}(k,r))$$

It turns out that this correspondence does not preserve the top degree part, hence does not give what we want.

Modified correspondence, which is more natural in symplectic geometry:

conormal bundle to $Q(n,r)$
lagrangian subvariety in the product

$$\begin{tikzcd}[column sep=small] & \{ 0 \subset \overset{k-1}{S_1} \subset \overset{k}{S_2} \subset \mathbb{C}^r,\ z \in \mathrm{Hom}(\mathbb{C}^r/S_2, S_1) \} = P(k,r) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ T^*\mathrm{Grass}(k-1,r) & & T^*\mathrm{Grass}(k,r) \end{tikzcd}$$

$z_1 \in \mathrm{Hom}(\mathbb{C}^r/S_1, S_1)$ $\qquad\qquad$ $z_2 \in \mathrm{Hom}(\mathbb{C}^r/S_2, S_2)$

Th [Ginzburg]

$e = \sum_k (-1)^{r-k} p_{1*} p_2^*$, $f = \sum_k (-1)^{k-1} p_{2*} p_1^*$ define the $(r+1)$-dimensional representation of sl_2 satisfying $v_k = [\mathrm{Grass}(k,r)]$.

Remarks

(1) Poincaré duality $H_*(T^*\mathrm{Grass}(k,r)) \cong H_c^{4d_2-*}(T^*\mathrm{Grass}(k,r))$

$d_2 = \dim_{\mathbb{C}} \mathrm{Grass}(k,r)$

$H_*(\mathcal{B}(k,r)) \cong H_c^{2(d_1+d_2)-*}(\mathcal{B}(k,r))$

$d_1 = \dim_{\mathbb{C}} \mathrm{Grass}(k-1,r)$

$$p_{1*}p_2^* : H_{2d_2}(T^*\mathrm{Grass}(k,r)) \longrightarrow H_{\underset{\displaystyle \overset{\shortparallel}{2d_1}}{2(d_1+d_2)-2d_2}}(T^*\mathrm{Grass}(k-1,r))$$

$\therefore$ top degree is preserved.

(2) p_1, p_2 : proper $\Longrightarrow$ p_1^*, p_2^* : well-defined on cohomology with compact support

Core of the proof is to recover coefficients $\begin{cases} f v_k = \underline{(k+1)} v_{k+1} \\ e v_k = \underline{(r-k+1)} v_{k-1} \end{cases}$

These appear as Euler numbers of fibers of projections

($k \to k+1$ for f)

$$\begin{tikzcd}[column sep=small] & \{ S_1^{k-1} \subset S_2^{k} \subset \mathbb{C}^r \} =: Q(k,r) \arrow[dl, "g_1"'] \arrow[dr, "g_2"] & \\ \mathrm{Grass}(k-1,r) & & \mathrm{Grass}(k,r) \end{tikzcd}$$

fiber of $g_1 \cong \mathrm{Grass}(1, \mathbb{C}^r/S_1) \cong \mathbb{P}^{r-k}$ Euler number $= r-k+1$

$\mathbb{C}^r/S_1$: $(r-k+1)$ dim

fiber of $g_2 \cong \mathrm{Grass}(k-1, S_2) \cong \mathbb{P}^{k-1}$ Euler number $= k$

S_2: k dim

Why Euler numbers?

Intersection theory of (co)homology group $\Rightarrow$ Euler class of the normal bundle to a submanifold.

symplectic vector space M $\cong$ Lagrangian subspace $L \oplus L^*$ $\therefore M/L \cong L^*$

Finally, the sign's come from $e(E^*) \cong (-1)^{\mathrm{rk}E} e(E)$ for cpx vector b'dle E.

This construction can be generalised to quiver varieties:

$$\mathfrak{P}_i(V,W) = \{ (B, a, b, \underset{\text{codim}=1}{S \subset V_i}) \mid V_i \supset S \xleftarrow{B_h} V_j,\ W_i \xrightarrow{a_i} S \}$$

$$\mathfrak{M}(V',W) \xleftarrow{p_1} \mathfrak{P}_i(V,W) \xrightarrow{p_2} \mathfrak{M}(V,W)$$

$V' \oplus \underset{\uparrow \text{ at vertex } i}{\mathbb{C}} = V$

p_1 : restriction to S

p_2 : forgetting S

- $\mathfrak{P}_i(V,W)$ is a lagrangian submanifold of $\mathfrak{M}(V',W) \times \mathfrak{M}(V,W)$ (smooth)
- fibers of p_1, p_2 are Grassmannian manifolds.

V, W give weights by

$$W \longrightarrow \lambda = \sum_i \dim W_i \cdot \Lambda_i$$

$$V \longrightarrow \mu = \lambda - \sum_i \dim V_i \, \alpha_i$$

$\Lambda_i = i^{\text{th}}$ fundamental weight

$\langle h_j, \Lambda_i \rangle = \delta_{ij}$

α_i : simple root

Th [N, 1998]

(1) $e_i = \sum (\pm)\, p_{1*} p_2^*$, $f_i = \sum (\pm)\, p_{2*} p_1^*$ define a representation of $\mathfrak{g}_Q$ on $\bigoplus H_{\dim \mathfrak{M}(v,w)}(\mathfrak{M}(v,w))$

summand $H_{\dim}(\mathfrak{M}(v,w))$ is the weight $= \mu$ subspace.

(2) It is the integrable highest weight (hence irreducible) representation $V(\lambda)$ with highest weight $\lambda = \sum \dim W_i \Lambda_i$

s.t. highest weight vector $= [\underbrace{\mathfrak{M}(0,w)}_{= pt}]$

Remarks (1) $H_{\dim \mathfrak{M}(v,w)}(\mathfrak{M}(v,w)) \cong H_{\underbrace{2\dim \mathfrak{L}(v,w)}_{top}}(\mathfrak{L}(v,w))$

has a base given by irreducible components of $\mathfrak{L}(v,w)$.

(2) Example in p. 10
$$\begin{tikzcd} \mathbb{C} \arrow[r, shift left] \arrow[d, shift left] & \mathbb{C} \arrow[l, shift left] \arrow[d, shift left] \\ \mathbb{C} \arrow[u, shift left] & \mathbb{C} \arrow[u, shift left] \end{tikzcd}$$
gives weight $= 0$ subspace in the adjoint representation of $\mathfrak{sl}_3$

(3) The proof of the defining relations (e.g. $[e_i, f_i] = h_i$) is somewhat local at vertex i. $\Rightarrow$ not significantly different from type A_1 case

(4) For the proof of the highest weight property (irreducibility), We use inductive construction of irreducible components of $\mathcal{L}(V, W)$ in (1).

$\rightsquigarrow$ Kashiwara's crystal structure on $\bigsqcup_V \operatorname{Irr} \mathcal{L}(V, W)$.

PART 3. Further Topics

(I). Allow $\circ\!\circlearrowleft$

Example Jordan quiver $\circ\!\circlearrowleft$ one vertex, one edge

$$\begin{array}{c} B_1 \circlearrowright \mathbb{C}^k \circlearrowleft B_2 \\ b \downarrow\uparrow a \\ \mathbb{C}^{r=1} \end{array} \qquad \mathfrak{M}(v,w) = \mathrm{Hilb}^k(\mathbb{C}^2)$$

Hilbert scheme of k points in $\mathbb{C}^2$

One can construct a representation of Heisenberg algebra in a similar way

Reference. N. Lectures on Hilbert schemes of points on surfaces, AMS

(II) Tensor product representations

We assume $W = W^1 \oplus W^2$ decomposition of Q_0-graded vector space

Consider $bBB\cdots Ba : W \longrightarrow W$ and $B\cdots B : V \longrightarrow V$

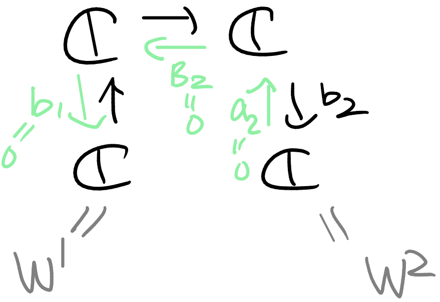
$$\mathfrak{J}(V : W^1, W^2) \overset{\text{def}}{=} \{ [B,a,b] \in \mathfrak{M}(V,W) \mid bB\cdots Ba(W^2) = 0 \}$$

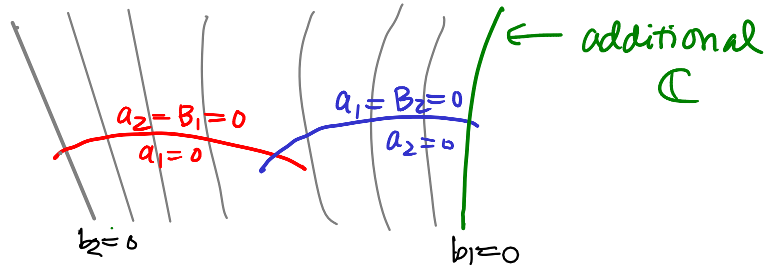
$$\mathrm{Im}\, bB\cdots Ba \subset W^2$$

$$\mathrm{tr}(B\cdots B) = 0$$

lagrangian subvariety

Example in p. 10





(Borel-Moore homology)

Th

$$\bigoplus_V H^{BM}_{\dim \mathfrak{M}(V,W)}(\mathfrak{T}(V,W)) \quad \left(= \bigoplus H^{\dim \mathfrak{M}(V,W)}(\mathfrak{M}(V,W), \mathfrak{M}(V,W) \setminus \mathfrak{T}(V,W)) \right)$$

$\cong$ tensor product representation $V(\lambda^1) \otimes V(\lambda^2)$

$$\lambda^1 = \sum \dim W^1_i \Lambda_i, \quad \lambda^2 = \sum \dim W^2_i \Lambda_i$$

The example above gives the weight $= 0$ subspace in $\mathbb{C}^3 \otimes (\mathbb{C}^3)^*$ of $\mathfrak{sl}_3$ representation

Note $V(\lambda^1) \otimes V(\lambda^2) \cong V(\lambda^2) \otimes V(\lambda^1)$, but $\mathfrak{T}(V; W^1, W^2) \neq \mathfrak{T}(V; W^2, W^1)$

$$v \otimes w \mapsto w \otimes v$$

Maulik-Okounkov gave an isomorphism $H^{BM}_{\dim \mathfrak{M}(V,W)}(\mathfrak{T}(V; W^1, W^2)) \cong H^{BM}_{\dim \mathfrak{M}(V,W)}(\mathfrak{T}(V; W^2, W^1))$
2019 (stable envelope)

III Quantum loop algebra

$\mathfrak{g}_Q$ = Kac–Moody Lie algebra

$L\mathfrak{g}_Q = \mathfrak{g}_Q[z, z^{-1}]$: loop algebra of $\mathfrak{g}_Q$

$U_q(L\mathfrak{g}_Q)$ = quantum loop algebra
(defined as a straightforward generalization of Drinfeld "new" realization of $U_q(L\mathfrak{g}_Q)$ for $\mathfrak{g}_Q$ = finite dim'l cpx simple Lie algebra)

eg. $\mathfrak{g}_Q$ = affine Lie algebra $\Longrightarrow$ $U_q(L\mathfrak{g}_Q)$: quantum toroidal algebra

$U_q^{\mathbb{Z}}(L\mathfrak{g}_Q)$: $\mathbb{Z}[q, q^{-1}]$-form

$K_{\mathbb{C}^\times \times G_W}(\mathcal{L}(V, W))$: equivariant K-group = Grothendieck group of category of equivariant coherent sheaves

<u>Th</u> [N, 2000]

$\bigoplus_V K_{\mathbb{C}^\times \times G_W}(\mathcal{L}(V, W))$ is a representation of $U_q^{\mathbb{Z}}(L\mathfrak{g}_Q)$

The proof used the presentation of $U_q(L\mathfrak{g}_Q)$

Maulik-Okounkov, Okounkov-Smirnov introduced a new (and better) approach.

They constructed an isomorphism

$$\bigoplus_{V=V^1\oplus V^2} K_{eq}(\mathfrak{L}(V^1,W^1)) \otimes K_{eq}(\mathfrak{L}(V^2,W^2)) \overset{st}{\cong} K_{eq}(\mathfrak{Z}(V;W^1,W^2))$$

by (K-theoretic) stable envelope.

On the other hand $\mathfrak{L}(V^1,W^1) \times \mathfrak{L}(V^2,W^2) \hookrightarrow \mathfrak{Z}(V;W^1,W^2)$ gives

$$\bigoplus_{V=V^1\oplus V^2} K_{eq}(\mathfrak{L}(V^1,W^1)) \otimes K_{eq}(\mathfrak{L}(V^2,W^2)) \otimes \mathrm{Frac} \overset{\text{naive}}{\cong} K_{eq}(\mathfrak{Z}(V;W^1,W^2)) \otimes \mathrm{Frac}$$

Then $\text{naive}^{-1} \circ \text{stab} : \bigoplus_{V=V^1\oplus V^2} K_{eq}(\mathfrak{L}(V^1,W^1)) \otimes K_{eq}(\mathfrak{L}(V^2,W^2)) \otimes \mathrm{Frac} \circlearrowleft$ isom.

This gives a geometric construction of R-matrix

$$V(\lambda^1) \otimes V(\lambda^2) \otimes \mathrm{Frac} \circlearrowleft$$

$U_q(L\mathfrak{g}_Q)$ (a larger algebra more generally) is reconstructed from the R-matrix.